OLIN COLLEGE

Trifilar Pendulum: Measurement and Error Analysis

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1 ABSTRACT

Trifilar pendulums are easy-to-make devices that can be used to experimentally derive the moment of inertia of an object about its center of mass. However, the equations of motion used to obtain this datum make use of several simplifying assumptions. In this project, the full frequency response of the system is recorded and the amplitude decay over time is fit to a simple viscous damping model using a custom quadrature encoder and MATLAB, respectively. This model is then used to calculate damping ratio equation 1.2, which is used to adjust oscillatory period in an attempt to make moment of inertia calculation more accurate. The equation that determines the moment of inertia of the object being measured is:

$$I_{object} = \frac{R^2 g \tau^2}{4\pi^2 L}(M_{platform} + M_{object}) - I_{platform} \tag{1.1}$$

The equation that determines the damping factor of that measurement is:

$$\zeta = \frac{-c * \tau_d}{\sqrt{c^2 * \tau_2^2 + 4\pi^2}} \tag{1.2}$$

The results of this project implied that, for the system tested, forcing the damping ratio to fit a viscous damping model based on amplitude decay does not change the error percentage of moment of inertia calculation by a statistically significant amount. Subsequently, error calculations were performed which concluded that measurement error is worse when the ratio of mass to moment of inertia is high.

2 BACKGROUND

This project involved building and testing a trifilar pendulum. At its simplest, a trifilar pendulum is a platform rotating about a point via three evenly-spaced vertical strings attached to the outsides of the platform. When the disk is rotated, the strings are pulled to an angle and the disk is lifted very slightly. Then when it is released, the angled strings put a torque on the platform that turns it in the other direction, twisting it up a small distance on the other side of its equilibrium. Then it oscillates for a period of time as determined by damping. The period of this oscillation is proportional to the moment of inertia, such that the moment of inertia can be determined through measuring the period of oscillation.

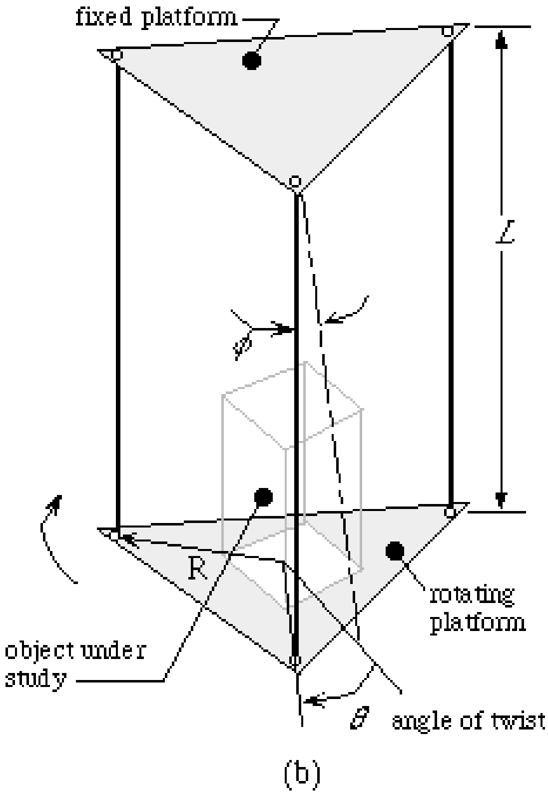

Figure 2.1: A basic trifilar pendulum diagram.
http://www.me.utexas.edu/ me244L/labs/filar/filaroverview.html

These measurement systems are commonly used for determining moments of inertia in many fields such as the automotive or aerospace industries. In these situations, an object is put into a special frame such that it can be suspended about many different axes and then entire moment of inertia tensor can be measured. [4] From the equations of motion, a relationship between the period of oscillation, mass of the system, and moment of inertia about the central axis can be made, shown below. This relationship can be used to measure the moment of inertia about the center of mass of any item placed on the disk.

$$I_{object} = \frac{R^2 g\tau^2}{4\pi^2 L}(M_{platform} + M_{object}) - I_{platform} \tag{2.1}$$

Where:
I refers to moment of inertia through center of mass about the z-axis of the system
R is the distance from the center of the disk to each string
g is acceleration due to gravity (approximated to by 9.8 m/s2)
τ is the period of oscillation
L is the length of the strings
m refers to mass
Reaching the general equation of motion of the disk would be quite complicated as the motion is three dimensional and non-linear. However, the motion of the disk can be simplified to two dimensions by using small angle approximation and a very large ratio between the length of the strings and the radius of the disk. These approximations are commonly accepted as reasonable and are used in this project. [2] [3] The derivation of the equations of motion for this project was based on the assignment detailed in a class design project from Brown University[1].

3 Equation of Motion Derivation

This derivation determines the moment of inertia of an object from the period of its oscillation on a trifilar pendulum. There are several important assumptions to note. First, the small angle approximation is used for both $sin(\theta) = \theta$ and $cos(\theta) = 1$. In addition, the vertical motion of the platform is assumed to be zero, due to the small angle approximation.

3.1 Diagram

This diagram is taken from a Brown University Assignment on finding the equations of motion of a trifiar pendulum.

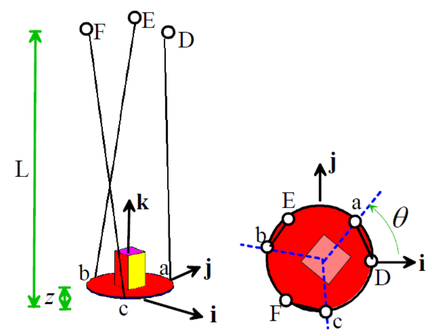

Figure 3.1: A basic trifilar pendulum diagram [1]

3.2 Location of Fixture Points

The derivation begins by finding the locations of the fixture points of the ends of strings. The reference frame used throughout this derivation is centered at the center of the platform in its static equilibrium with the z axis pointed upwards and the x axis pointed towards connection point A.

$$Rd = \begin{bmatrix} R \\ 0 \\ L \end{bmatrix} \tag{3.1}$$

$$Re = \begin{bmatrix} -.5R \\ .866R \\ L \end{bmatrix} \tag{3.2}$$

$$Rf = \begin{bmatrix} -.5R \\ -.866R \\ L \end{bmatrix} \tag{3.3}$$

3.3 Location of Fixture Points on the Disk

$$Rd = \begin{bmatrix} R * cos(\theta) \\ R * sin(\theta) \\ z \end{bmatrix} \tag{3.4}$$

$$Re = \begin{bmatrix} R * cos(\theta + \frac{2\pi}{3}) \\ R * sin(\theta + \frac{2\pi}{3}) \\ z \end{bmatrix} \tag{3.5}$$

$$Rf = \begin{bmatrix} R * cos(\theta - \frac{2\pi}{3}) \\ R * sin(\theta - \frac{2\pi}{3}) \\ z \end{bmatrix} \tag{3.6}$$

3.4 Unit Vectors along Strings

The unit vector along each string is equal to the position of the upper fixture point minus the disk fixture point divided by the length of the string.

$$S1 = \frac{Rd - Ra}{L} = \begin{bmatrix} R - R * cos(\theta) \\ -R * sin(\theta) \\ L - z \end{bmatrix} * \frac{1}{L} \tag{3.7}$$

$$S2 = \frac{Re - Rb}{L} = \begin{bmatrix} -.5R - Rcos(\theta + \frac{2\pi}{3}) \\ .866R - Rsin(\theta + \frac{2\pi}{3}) \\ L - z \end{bmatrix} * \frac{1}{L} \tag{3.8}$$

$$S3 = \frac{Rf - Rc}{L} = \begin{bmatrix} -.5R - Rcos(\theta - \frac{2\pi}{3}) \\ -.866R - Rsin(\theta - \frac{2\pi}{3}) \\ L - z \end{bmatrix} * \frac{1}{L} \tag{3.9}$$

3.5 Small Angle Assumptions

At this point, certain assumptions must be made to linearize the model. These all stem from the small angle approximation of $sin(\theta)$ and $cos(\theta)$. The final result of this is that the vertical travel of the pendulum can be assumed to be zero.

$$z = 0 \tag{3.10}$$

From this we know that:

$$\ddot{z} = 0 \tag{3.11}$$

Therefore, the pendulum is not accelerating vertically and the equation F = ma can be used to determine the tension in the strings.

$$3T - mg = 0 \tag{3.12}$$

$$T = mg/3 \tag{3.13}$$

At this point we can use an Euler equation, as we know the direction of the moment arms, the direction of the tension force and the magnitude of the tension force.

$$I\ddot{\theta} = T\frac{R_a \times (R_d - R_a)}{L} + T\frac{R_b \times (R_e - R_b)}{L} + T\frac{R_c \times (R_f - R_c)}{L} \tag{3.14}$$

When these cross products are evaluated, the result is that:

$$I\ddot{\theta} + \frac{mgR^2}{L}\theta = 0 \tag{3.15}$$

When this equation is solved the final result for moment of inertia is that

$$I_{object} = \frac{R^2 g\tau^2}{4\pi^2 L}(M_{platform} + M_{object}) - I_{platform} \tag{3.16}$$

4 Learning Objectives

The goal of this project was to build a trifilar pendulum and determine how much the accuracy of the resultant moment of inertia calculation could be improved by taking damping into account. Another goal was to automate our calculation as much as possible by making a custom quadrature encoder that recorded data from each run, sent it to an Arduino Uno, and then had a computer process the data in MATLAB to yield damping ratio. This automation makes it very easy to run multiple tests on varying objects and decreases the labor needed to analyze each run. A final goal was to run error analysis on the data and draw conclusions about the accuracy of the experimental setup overall.

5 System Model

This model takes into account length of the strings, radius of the platform, moment of inertia of the platform, and the moment of inertia of the test object. It specifically neglects displacement on the z-axis, makes small-angle approximations for trigonometric functions of θ, and ignores all sources of damping (friction in strings, air resistance, and, in our case, friction from our stabilization peg). Damping was calculated but was determined to be statistically insignificant.

5.1 Mechanical

The pendulum used was built from relatively low-cost materials. A top and bottom base support were measured and cut from MDF as truncated equilateral triangles (with an extension on the bottom base to support an encoder). The support beams are pre-cut wood banister beams, chosen for their straightness, consistent length and square ends. The supporting wire is Kevlar spear fishing wire. This cable was selected because of its low stretch. Our model does not account for stretch in the strings and this string reduced error from that source. This wire may have added more frictional error than expected due to its large diameter and stiffness, which will be discussed later in results. The plate is .118" laser cut acrylic. A steel pin was placed through the center of the plate and into a delrin block to horizontally constrain the plate's motion to rotation about a fixed point. We determined the moment of inertia of or plate very precisely through first creating the system in SolidWorks, and then creating a custom material to ensure that the density of our acrylic was being accurately accounted for.

Figure 5.1: Mechanical System

6 Electrical System

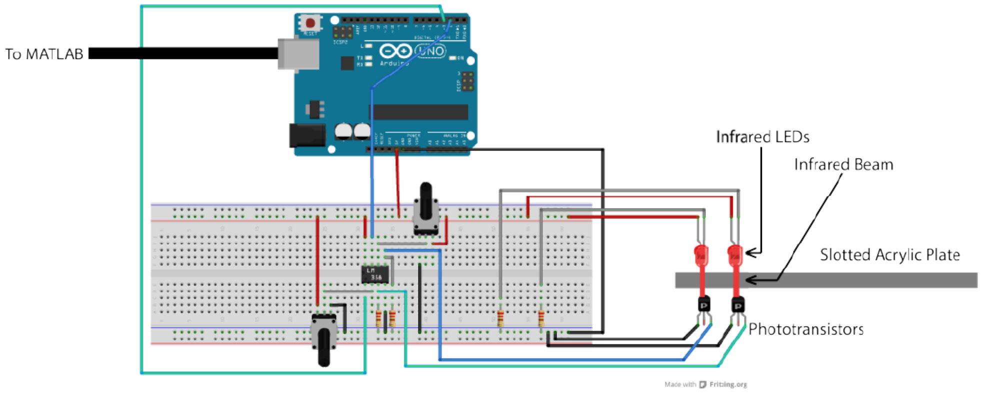

Figure 6.1: The circuit for the measurement system of the pendulum. Two IR leds are positioned such that they will return four different counts over a rotation of 3 degrees. The information from the diodes is send to an op-amp that sets a cutoff voltage above which the signal to the arduino is 5v and below which it is zero. The arduino then logs the counts and determines what direction the system is moving in and how many counts have passed. This data is then sent to matlab which plots the position of the system and determines the period.

Measurements were taken from the spinning disk by a quadrature encoder made from two well-placed LED-photo-transistor pairs mounted on a custom-milled aluminum block that allowed the disk to spin in between the photo-transistors and LEDs. The disk was laser cut from acrylic with precisely-designed slots that allowed the encoder to detect direction and amplitude of oscillation as the disk rotated over time. The LED-phototransistor pairs transmitted signal to an Arduino Uno corresponding to whether or not the LED-to-photo-transistor beam was blocked by the acrylic plate at a given instant. A simple amplification circuit was placed between these pairs and the Arduino so that the signal from the pairs would arrive as a well-defined square wave signaling that a pair was 'blocked' or 'unblocked'. The Arduino then processed the relative signals of the pairs to determine amplitude over time. These data were then transmitted to MATLAB, which plotted the data for verification and processed the numbers to calculate moment of inertia and error. This system was very effective in that data could be captured quickly and easily. It was also accurate to less than a degree. The only major issue with it was that when the disk spun too rapidly, the encoder would miss counts and give unreliable data. However, this only occurred when the disk was displaced by significantly more than an acceptable angle for small-angle assumptions. In this report, it is assumed that the encoder did not miss counts and that the data are accurate.

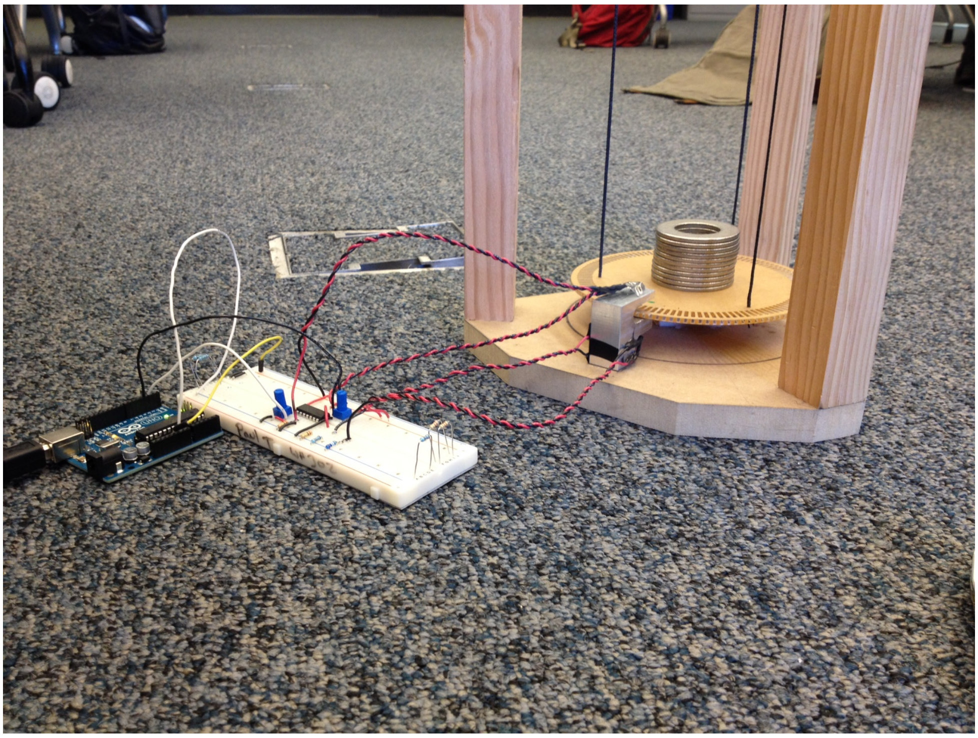

Figure 6.2: A close up image of the encoder and plate working together.

7 Results

Using the quadrature encoder and MATLAB processing, full frequency responses were extracted for analysis. MATLAB calculated time between peaks and averaged them to come up with period of oscillation. It also used the "peaks" function to trace the degradation of amplitude over time. By assuming that this decay fit the form $Xe^{-\zeta\omega_n t}$ (viscous damping model), the natural logarithm of that curve could be fit to a first order polynomial with the "polyfit" MATLAB function and a damping ratio could then by calculated by:

$$\zeta = \frac{-c * \tau_d}{\sqrt{c^2 * \tau_2^2 + 4\pi^2}} \tag{7.1}$$

Where:
c is the coefficient of the linear term produced by fitting the natural logarithm of the amplitude degradation to a line.
T_d is the measured (damped) period of oscillation.
ζ was then used to calculate the natural period of oscillation using:

$$\tau_n = \tau_d \sqrt{1 - \zeta^2} \tag{7.2}$$

To calculate results, we first plotted the data in MATLAB and then determined peaks. Finally, we determined the time between the peaks to find the final period of oscillation. An example of this raw data is shown below. Damping values were also calculated but were statistically insignificant so have not been heavily analyzed.

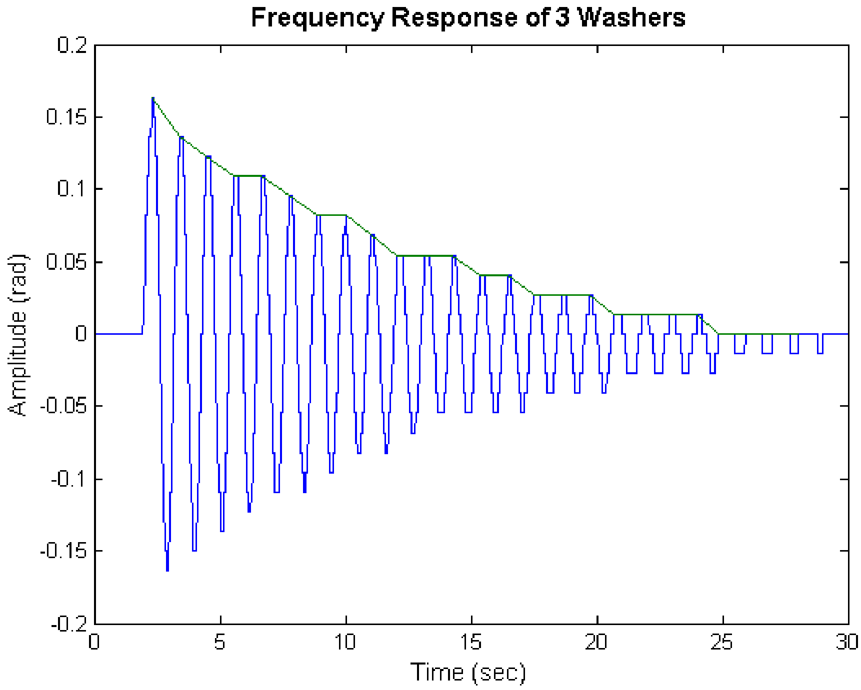

Figure 7.1: Frequency response with its corresponding amplitude decay curve. All runs looked similar to this, varying in frequency and specific decay curve.

Theoretical moments of inertia were calculated for several samples by massing each with scales and measuring geometry with digital calipers. Then, moments of inertia (MOI) were experimentally derived using the trifilar pendulum and error percentages were calculated using both the measured period of oscillation and the corrected period using ζ. Finally, a student's t-test was used to determine if the moment of inertia derived with the corrected period was significantly different from the moment of inertia derived with the original, damped period. In all cases, $p>.5$, so they are not significantly different. Therefore, in all following analysis, the damped frequency will be used for measurements. The tables produced are shown below:

Wood Block

Trial #	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.039839
1	6.0900E-05	6.1108E-05	6.0946E-05	0.3415%	0.0755%	
2		6.4850E-05	6.3940E-05	6.4860%	4.9918%	**mass/MOI**
3		6.0050E-05	5.8900E-05	-1.3957%	-3.2841%	654.1707718
4		6.1706E-05	6.1520E-05	1.3235%	1.0181%	
5		6.0616E-05	6.0414E-05	-0.4663%	-0.7980%	***t*** **-test (p)**
Averages		6.1666E-05	6.1144E-05	1.2578%	0.4007%	0.669296217

ABS block

Trial #	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.07113
1	7.6923E-05	7.6847E-05	7.6738E-05	-0.0983%	-0.2400%	
2		7.4789E-05	7.4669E-05	-2.7737%	-2.9297%	**mass/MOI**
3		7.5230E-05	7.5130E-05	-2.2004%	-2.3304%	924.6957331
4		7.8018E-05	7.7927E-05	1.4240%	1.3057%	
5		7.7799E-05	7.7700E-05	1.1393%	1.0104%	***t*** **-test (p)**
Averages		7.6537E-05	7.6433E-05	-0.5018%	-0.6368%	0.914062296

2 Wood Blocks

Trial #	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.079678
1	1.2180E-04	1.1497E-04	1.1200E-04	-5.6076%	-8.0497%	
2		1.2500E-04	1.2297E-04	2.6273%	0.9606%	**mass/MOI**
3		1.1648E-04	1.1618E-04	-4.3678%	-4.6141%	654.1707718
4		1.2349E-04	1.2329E-04	1.3875%	1.2233%	
5		1.2120E-04	1.2093E-04	-0.4926%	-0.7143%	***t*** **-test (p)**
Averages		1.2023E-04	1.1907E-04	-1.2906%	-2.2388%	0.703253945

Washers

# Washers	Theoretical MOI (kg*m^2)	Measured MOI (kg*m^2)		% Error		mass/unit (kg)
		Using Td	*Using Tn*	*Using Td*	*Using Tn*	0.031
0	0	2.5267E-06	2.3766E-06	N/A	N/A	
1	1.2880E-05	2.2062E-05	2.1965E-05	71.29%	70.54%	**Mass/MOI**
2	2.5760E-05	3.9323E-05	3.9292E-05	52.65%	52.53%	2406.832298
3	3.8640E-05	5.4671E-05	5.4595E-05	41.49%	41.29%	
4	5.1520E-05	6.4311E-05	6.4216E-05	24.83%	24.64%	***t*** **-test (p)**
5	6.4400E-05	8.0973E-05	8.0866E-05	25.73%	25.57%	0.983052405
6	7.7280E-05	9.4432E-05	9.4362E-05	22.19%	22.10%	
7	9.0160E-05	1.0265E-04	1.0256E-04	13.85%	13.76%	
8	1.0304E-04	1.1881E-04	1.1872E-04	15.30%	15.21%	
9	1.1592E-04	1.3308E-04	1.3294E-04	14.80%	14.69%	
10	1.2880E-04	1.4646E-04	1.4637E-04	13.71%	13.64%	

Figure 7.2: Data collected from trial runs. Theoretical moments of inertia, moments of inertia calculated from both τ_n and τ_d, errors and error factors are reported. The statistical significance of the difference between the data using τ_n and τ_d is reported. In no trial is the difference significant.

For analysis, the samples can be split into two categories: blocks and washers. The three test blocks pictured below are wood, ABS plastic, and wood from left to right. The rightmost sample consists of two identical wood blocks of the same mass and dimensions as the leftmost sample stacked vertically in the same orientation as the leftmost sample. These samples each ran through 5 trials on the trifilar pendulum and the calculated moments of inertia were recorded. These data showed that, based on a student t-test, using the corrected period Tn as calculated using a viscous damping model did not change the calculated moment of inertia by a statistically significant amount.

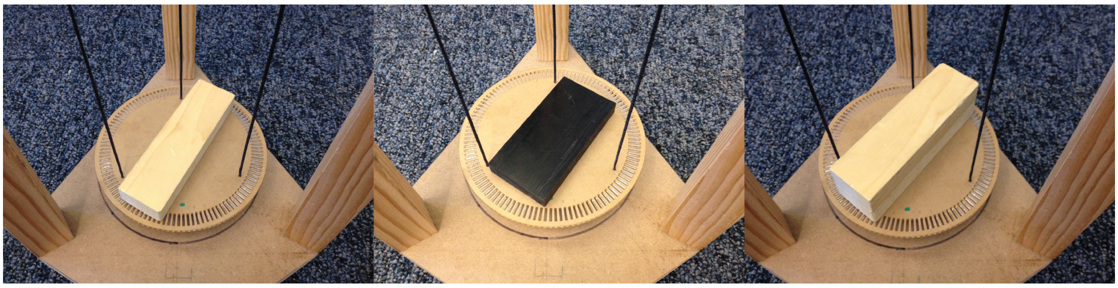

Figure 7.3: Block samples. From left to right: single block, ABS block, and two wooden blocks.

Large steel washers were also tested for several reasons. They can be modeled as thick hollow cylinders for theoretical MOI calculation, which is simple and well-understood formula. Washers are also very consistent in their geometries and are plentiful, so many of them could be gathered and treated as having the same moment of inertia (this was verified using a scale and digital calipers). Additionally, they can be stacked in a column which, in the theoretical MOI calculation, affects the mass term, but does not affect the mass to moment of inertia ratio.

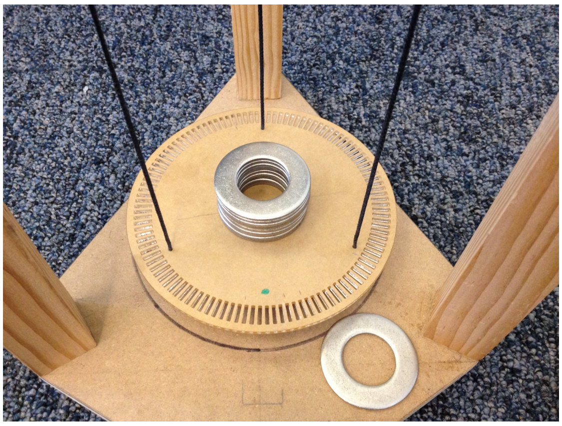

Figure 7.4: Washer arrangement for testing.

Washer MOI were measured by incrementing the number of washers stacked in the center of the platform up to ten washers and measuring MOI using the trifilar pendulum for each number of washers. As with the blocks, error percentage was calculated using both Td and Tn. As with the blocks, a student t-test showed no statistically significant change in MOI calculation between the two periods. It was observed, however, that error decreased dramatically as the number of washers in the stack increased until about seven washers were stacked, at which point error percentage leveled out. Since stacking the washers only increases the mass term of the theoretical MOI calculation, this relationship implies a correlation between accuracy of MOI measurement and mass of sample object. This idea is explored in more detail in the "Diagnosis and Discussion" section.

8 Error Analysis

Our system measured moments of inertia with consistently low errors in certain situations. When the ratio of mass to moment of inertia was low, the system was far more accurate. In addition, on very small masses, friction increased error drastically. However, this error was reduced through increasing the amount of mass used.

8.1 Measurement Error of Encoder

Our trifilar pendulum largely performed very well. It measured the moments of inertia of the blocks of wood and ABS with error margins of less than 3%. However, the error was significantly higher when we measured items such as steel washers. Rather than under 3%, it was always over 12% and as high as 71.29%. This error is potentially due to a combination of several factors.

As with many mechanical systems, precision in a trifilar pendulum is difficult to achieve. When objects have a low moment of inertia, period must be measured accurately to more than 1/100th of a second. In addition, the platform's moment of inertia must be determined very precisely such as not to offset the measure. Finally, an object must be perfectly centered on the platform such that its moment of inertia does not increase unreasonably and such that there is not excessive loading of the pin keeping the system rotating with 1 degree of freedom. If an object is off-center, then friction damping increases drastically due to the additional forces applied to the pin and the resulting friction. However, there are ways to avoid these errors. Objects must be chosen such that their ratio of mass of moment to inertia is as high as possible and they must be centered as well as possible on the platform. These specific error factors will be discussed in the following sections.

8.1.1 Derivation of Encoder Error

This derivation begins with determining the average tick lengths of a data set. In this case, it is done with data from 11 washers, however it should be the same for any sample, as it is dependent on the sensor, not the sample.

$$\tau_{actual} = \frac{d+c-a-b}{2} \tag{8.1}$$

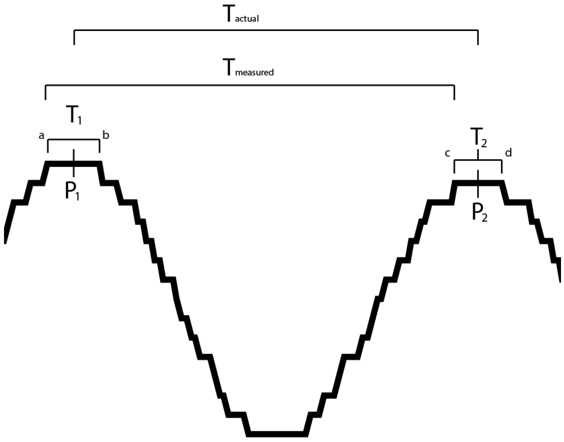

Figure 8.1: Variables for Encoder Error Calculation

$$\tau_{measured} = c - a \tag{8.2}$$

$$error = \pm(\tau_{actual} - \tau_{measured}) \tag{8.3}$$

$$error = \frac{(d-c)-(b-a)}{2} = \frac{T_1 - T_2}{2} \tag{8.4}$$

When this equation is expanded to get the average error, the worst period (Average plus two times standard deviation) is subtracted from the average period.

$$error_{avg} = \frac{T_{avg} - (T_{avg} - 2StDev)}{2} = StDev \tag{8.5}$$

Therefore, our measurement error from the encoder is twice the Standard Deviation of the tick length at the maximums. Using the data above error in period calculation is determined to be $\pm.07997s$, which is slightly larger than, but roughly equal to the deviations in measurements observed during testing.

8.1.2 Error Multiplication by Mass to Moment of inertia ratio

During our testing, it was observed that the error on measurements of objects that had a high $\frac{Mass}{MomentofInertia}$ ratio had significantly (p<.05) higher errors than objects that had a comparatively low one. This was especially observed in the differences between the errors recorded when measure the moments of inertia of the washers ($\frac{Mass}{MomentofInertia}$ = 2406.8) and the moments of inertia of wooden and ABS blocks (($\frac{Mass}{MomentofInertia}$ = 654.1 and 924.7, respectively). We determined that this change is due to the equation that is used. The derivation of the error scaled by this ratio is shown here.

$$error = \frac{I_{experimental} - I_{theoretical}}{I_{theoretical}} \tag{8.6}$$

From this point onward, the constants in the moment of inertia equation shall be referred to as C.

$$C = \frac{r^2 g}{4\pi^2 L} \tag{8.7}$$

$$error = \frac{(C\tau^2(M_o + M_p) - I_p) - I_t}{I_t} \tag{8.8}$$

The M_p term and the I_p term will cancel, resulting in:

$$error = \frac{C\tau^2 M_o}{I_t} - 1 \tag{8.9}$$

In this equation, as the ratio of $\frac{Mass}{MomentofInertia}$ increases, error increases proportionally.

8.1.3 Error due to friction

Error due to friction was not a focus of this study, and has therefore not been calculated. However, it is clear that there is a proportion of the error that is due to friction. In the washer data, as mass increases, error decreases, in a function that can be linked to $\frac{a}{x}$ where a is the force due to friction. This is due to the fact that as mass increases, the force due to the washers increases. However, the force due to friction always stays the same. Therefore, as force due to the washers increases, it eventually overwhelms the force due to friction, decreasing the error. However, friction is always present and therefore the error will never decline to zero.

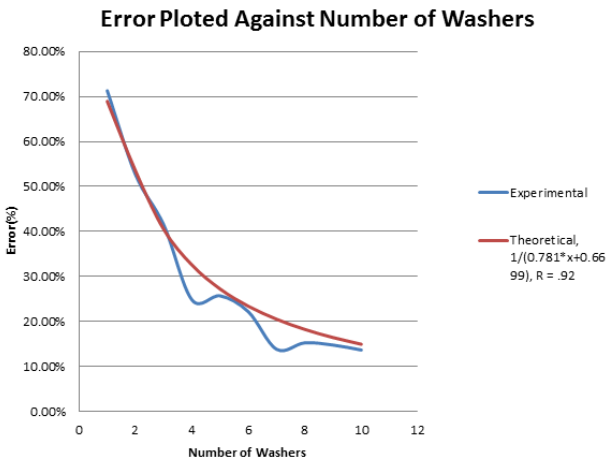

Figure 8.2: Error as number of washer increase. The data fits very well to a $\frac{a}{x}$ fitline, showing that friction is likely the cause of the excessive errors for low numbers of washers

8.1.4 Errors Combined

When all of the error calculations are combined the end result is shown below:

$$error = C(\tau_t \pm .08)^2 * \frac{M_o}{I_t} - 1 \tag{8.10}$$

Where $\tau_t \pm .08$ is the theoretical value that τ should be plus or minus the possible measurement error.

$$error = C(\tau_t^2 \pm .16\tau_t + .0064) * \frac{M_o}{I_t} - 1 \tag{8.11}$$

$C\tau_t^2 M_o$ is the theoretical moment of inertia for the object. Therefore it divided by I_t and the fraction is equal to 1.
The error is now:

$$error = C(\pm.16\tau_t + .0064) * \frac{M_o}{I_t} \tag{8.12}$$

However, this can be reduced further, because τ_t is mass and moment of inertia ratio by the following equation:

$$I_t = C * T_t^2 * M_o \tag{8.13}$$

$$T_i = \sqrt{\frac{1}{\frac{Mo}{I_t}C}} \tag{8.14}$$

Therefore, the final error calculation, with friction neglected, is:

$$error = C(\pm.16\sqrt{\frac{1}{\frac{Mo}{I_t}C}} + .0064) * \frac{M_o}{I_t} \tag{8.15}$$

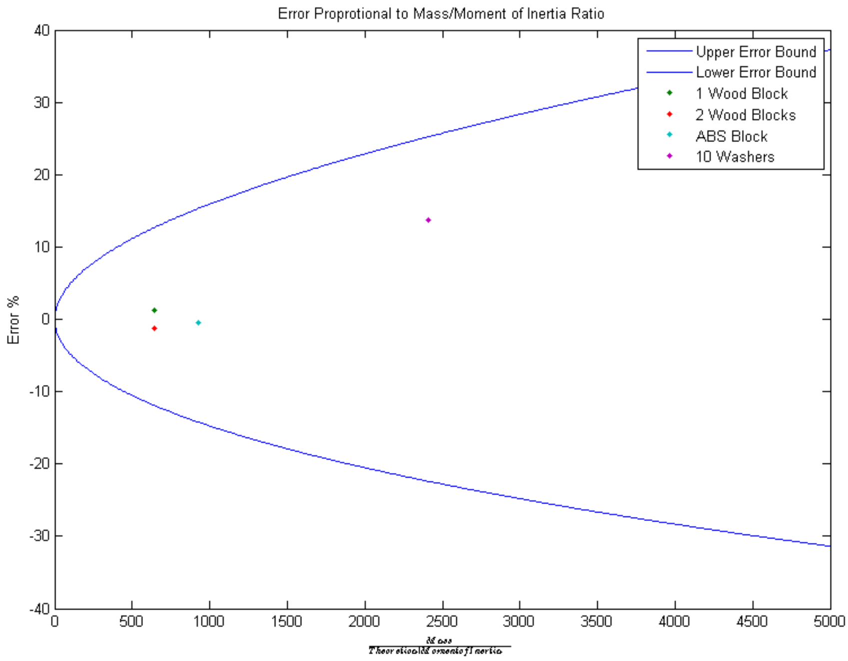

Figure 8.3: Error bounds plotted with sample errors from our measurements. The error used for the washer is the washer measurement with the largest number of washers. This should diminish the effects of friction upon this measurement. All of the measurements are reasonably within the error bounds. Most of our measurements are much better than the expected error bounds, but this is simply due to random chance. The fact that they are within the bounds is indicative that our error analysis is correct.

9 Damping Discussion

The damping on this system is a combination of linear damping due to friction and exponential damping due to air resistance. The original plan of this project was to determine the undamped, natural frequency of oscillation and then use that information to calculate a more accurate moment of inertia. However, when this calculation was performed, the difference between the two data sets was not statistically significant ($p>.5$), and in certain situations it increased the error. When friction damping was modeled and simulated, the differences in period were also not significant. Therefore, all of the final numbers used in this project are the damped frequency measure directly from our test setup.

In addition, the error on the calculation of damping was much greater than the error of the unprocessed measurement. Therefore, the inclusion of the natural frequency calculation would further increase error and therefore is undesirable.

10 Improvement

This model added sophistication to the commonly used basic trifilar pendulum setup by recording the full oscillatory response of the pendulum instead of just the period. This allowed analysis of the degradation in amplitude over time to account for damping in the system. One improvement that could have been made to increase accuracy is to increase the size of the entire system. While this would potentially increase error for smaller objects as we discovered with our washer measurements, it would allow the encoder higher resolution and would reduce the effects of friction of the stabilizing peg in the middle and the thickness of the support strings relative to the moment of the platform.

Another improvement to the system would be to make it more permanent. This would allow the pendulum to be on a consistently level surface and would allow the encoder to be fixed to that surface, allowing for better calibration of the system.

11 Reflection

The first lesson from this project was the importance of both accuracy and precision. Seemingly trivial factors like string width, levelness of the test surface, and how long the pendulum had sat without use all affected experimental results significantly. Especially when working with abstracted, heavily simplified math, it is important to make the experimental setup match the system's assumptions as best as possible.

Another lesson learned from this project is the importance peer review conducted as frequently as possible. Several delays in this project were caused by overlooking simple but fundamental errors in math, code, circuitry, etc. Minimizing these errors is important to reduce wasted time; catching an error quickly can save hours of debugging later.

12 Conclusion

In this project, a trifilar pendulum was built and outfitted with a custom quadrature encoder to detect the full frequency response of the system's oscillations. Sample blocks and larger washers were tested on this pendulum and error percentages between measured and theoretical moments of inertia were calculated to arrive at the following results:

1. Fitting the system to a viscous damping model by adjusting the oscillatory period based on decay of amplitude of oscillation over time does not yield a statistically significant change in the error percentage of the moment of inertia calculation of the test object.

2. Samples of lower mass-moment of inertia ratios resulted in lower errors.

13 Future Use

This project was a good application of the skills taught in Dynamics. It incorporated a design aspect and requires real-world problem assessment that would otherwise not be taught in

the course. In the time provided, however, the project was potentially too open-ended. More might have been gained from a more specific assignment as proposed below with a pre-built tifilar pendulum. It would involve less construction time and thus could focus more on the dynamics behind the system, while still including some debugging and real-world issues.
Proposed Trifilar Pendulum Assignment:

1. Derive the equation of motion to describe the rotation of the trifilar pendulum disk over time. Assume that the initial angular offset of the pendulum is small and that the displacement of the disk along the z-axis is negligible. What must be true about the relative dimensions of the disk and the strings for the latter assumption to be true?

2. Use this equation to find a relationship between moment of inertia of the plate and the parameter of the system (mass, disk radius, string length, etc.). How would you adapt this equation to find the moment of inertia of an object placed on top of the disk with its center of mass located directly above the disk's center?

3. Use the trifilar pendulum in the classroom and a stopwatch to measure the period of an object that you have the theoretical moment of inertia for (either via CAD or calculation). Using the relationship found above, how close is your experimental measurement to your theoretical moment of inertia? What techniques can you try to improve accuracy?

4. For each of these techniques, calculate percent error of experimental moment of inertia to theoretical. From these, what factors seem most important in reducing experimental error? Can you connect this error to the ratio of mass to moment of inertia of your test object(s)?

14 References

References

[1] A Trifilar Pendulum to Measure Mass Moments of Inertia. Brown.edu. Brown University, n.d. Web. 14 Dec. 2013.

[2] Du Bois, J. L., N. A. J. Lieven, and S. Adhikari. "Error Analysis in Trifilar Inertia Measurements." Experimental Mechanics 49.4 (2009): 533-40. Print.

[3] Moment of Inertia Test Lab. N.p., 24 Sept. 2003. Web. 14 Dec. 2013

[4] Williams, Huw. "Measuring the Inertia Tensor." Lecture. IMA Mathematics 2007 Conference. 26 Apr. 2007. Web. 14 Dec. 2013.

15 APENDIX A: ARDUINO DATALOGING CODE

```
/*
  Button

 Turns on and off a light emitting diode(LED) connected to digital
 pin 13, when pressing a pushbutton attached to pin 2.


 The circuit:
 * LED attached from pin 13 to ground
 * pushbutton attached to pin 2 from +5V
 * 10K resistor attached to pin 2 from ground

 * Note: on most Arduinos there is already an LED on the board
 attached to pin 13.


 created 2005
 by DojoDave <http://www.0j0.org>
 modified 30 Aug 2011
 by Tom Igoe

 This example code is in the public domain.

 http://www.arduino.cc/en/Tutorial/Button
 */

// constants won't change. They're used here to
// set pin numbers:
const int LED1pin = 2;
const int LED2pin = 3;// the number of the pushbutton pin
const int ledPin =  13;      // the number of the LED pin

// variables will change:
int LED1 = 0;
int LED2 = 0;
int time = 0;
int STATE;
int STATEold = 0;
int dir;
int DIREC;
int count;
```

```
void setup() {
  // initialize the LED pin as an output:
  pinMode(ledPin, OUTPUT);
  // initialize the pushbutton pin as an input:
  pinMode(LED1, INPUT);
  pinMode(LED2, INPUT);
  Serial.begin(9600);
}

void loop(){
  // read the state of the pushbutton value:
  LED1 = digitalRead(LED1pin);
  LED2 = digitalRead(LED2pin);

  if(LED1==HIGH){
      if(LED2==HIGH){
        STATE = 1;
      }
      else{
        STATE = 2;
      }
  }
  else{
      if(LED2==HIGH){
        STATE = 4;
      }
      else{
        STATE = 3;
      }
  }

DIREC =  STATE-STATEold;

if(DIREC==-1||DIREC==3){
  dir = 1;
}

if(DIREC==1||DIREC==-3){
  dir = -1;
}
else{
```

```
  dir = dir;
}

if(STATE!=STATEold){
  count = count + dir;
}

Serial.print(millis());
Serial.print(" ");
Serial.println(count);
//Serial.print(" ");
//Serial.print(STATE);
//Serial.print(" ");
//Serial.println(dir);


STATEold = STATE;


}
```

16 APPENDIX B: MATLAB DATALOGGING CODE

```
% Communications MatLab <--> Arduino
% Matlab file 1 for use with Arduino file 1
clc;
clf
clear all;

areaDensity = 0.0035928916
Density = areaDensity/2.9972

format long g;
numcyc=20;
t=[];
v=[];

s1 = serial('COM3');    % define serial port
s1.BaudRate=9600;              % define baud rate
set(s1, 'terminator', 'LF');   % define the terminator


resolution = 3

for i = 1:600;
    x(i) = i;
end;


%----------------------------------------------------------
%Starting to pull data


fopen(s1);
try                            % use try catch to ensure fclose
                               % signal the arduino to start collection
w=fscanf(s1,'%s');             % must define the input % d or %s, etc.
if (w=='A');
    display(['Collecting data']);
    fprintf(s1,'%s/n','A');    % establishContact just wants
                               % something in the buffer
end

i=0;
```

```
t0=tic;
intermediate(1) = 0;    %Initialize intermediate as 0
intermediate(2) = 0;

%%
while (i<(numcyc*100)); %While sensor is less
    %than 180 degrees yaw and 150 degrees pitch, letting it run a full
    %cycle
    clear v
    i=i+1; %increase index
    t(i)=toc(t0);
    t(i)=t(i)-t(1);

    v=fscanf(s1,'%c');        % must define the input % d or %s, etc.

    intermediate = sscanf(v,'%g') %intermediate values pulled from arduino
    a(i,1) = intermediate(1);     %Yaw
    a(i,2) = intermediate(2);     %Pitch

end


catch me;
    fclose(s1);                   %Close serial port s1


end
fclose(s1);
a(:,2) =  a(:,2)*.782*pi/180;
a(:,2) = a(:,2)-a(end,2);

plot(a(:,1)/1000,a(:,2))

[pks,locs] = findpeaks(a(:,2));
dsize = length(pks);

time = a(locs,1)/1000;

hold all
plot(time,pks,'.')


logpks = log(pks);
plot(time,logpks)
```

```
for i = 2:dsize-6
    pks2(i-1) = pks(i);
    time2(i-1) = time(i);
end
fit = polyfit(time2,log(pks2),1);




%%

pertot = 0;
for i=2:dsize
    per(i) = time(i)-time(i-1);
    pertot = pertot + per(i);
end

avgper = pertot/(dsize-1)

%%
% %finding period using fzero
%
% zeros = find(a(:,2)==0);
% timez = a(zeros,1);
% dsizez = length(timez);
% pertotz = 0;
% for i=2:dsizez
%     perz(i) = timez(i)-timez(i-1);
%     pertotz = pertotz + perz(i);
% end
%
% avgperz = pertotz/(dsizez-1)
%
% Tz = avgperz

%%
%damping
c = fit(1);
T = avgper;
zeta = (-c.*T)./sqrt(c.^2.*T.^2+4*pi^2);

Wd = 2*pi/avgper
```

```
Wn  = Wd/(sqrt(1-zeta^2));

Tn = 2*pi/Wn;

r = .05715;
g = 9.8;
l = .88185;
Mp = .0583;
Ip = 1.71655*10^-4; %as found by SOLIDWORKS
Ip2 =  .000158%1.0765e-04; %Version 2        6.4088*10^-5 %As derived experimentally

%Washer Masses (kg)
W1 = .02842;
W2 = .02853;
W3 = .02849;
W4 = .02848;
W5 = .02671;

TMOM =    (3.832 +   3.842 +    3.834 +   3.834 +   3.597 ) * 10^-5;

num = 10;
Mo = .0311*num%.039839*2%.0311*num;%W4+W3+W2+W1+W5 %.0311*num;% W1+W2%

MOM = (r^2*9.8*T^2/(4*pi^2*.8818)*(Mo+Mp))-.000114143 %(((r^2*g*T^2)/(4*pi^2*l))*(Mp+Mo)

TMOMBig = 0.00001288*num%6.09e-5*2%5.174*10^-5*num; %(3.834+3.834+3.842+3.597+3.832)*10^-


%Small Washer Moments%  1.0e-05 *of Inertia
%
%      3.832 (W1)
%      3.842 (W2)
%      3.834 (W3)
%      3.834 (W4)
%      3.597 (W5)

EPC = ((TMOMBig-MOM)/TMOMBig)*100;

nTME = [num;TMOMBig;MOM;EPC];

%Td -Tn comparison
TdTn = [T;Tn]

MOMn = r^2*9.8*Tn^2/(4*pi^2*.8818)*(Mo+Mp)-.000114143; %
```

```
EPCn = ((TMOMBig - MOMn)/TMOMBig)*100;

errors = [EPC;EPCn]
MOM
```

17 Appendix C: MATLAB Error Analysis Code

```
%Making PColor of error
clear all
r = .05715;
g = 9.8;
l = .88185;
Mp = .0583;
Ip = 1.71655*10^-4; %as found by SOLIDWORKS
Ip2 =  .000158%1.0765e-04; %Version 2


C = r^2*g/(4*pi^2*l);
% for i = 1:50;
%     for j = 1:500;
%         t(i,j) = i/25;
%
%         Rat(i,j) = j*10;
%         error1(i,j) = (C.*(.16.*t(i,j)+.0064).*Rat(i,j)-1).*100;
%         error2(i,j) = (C.*(-.16.*t(i,j)+.0064).*Rat(i,j)-1).*100;
%
%
%     end
% end
Ratio = linspace(1,5000,500)
t = sqrt(1./(Ratio*C))

error1 = ((C.*(.16.*t+.0064).*Ratio)).*100;
error2 = ((C.*(-.16.*t+.0064).*Ratio)).*100;

%figure(3)
%clf(3)
%pcolor(t,Rat,error1);
%figure(4)
%clf(4)
%pcolor(t,Rat,error2);
figure(5)
clf(5)
plot(Ratio,error1)
hold on
plot(Ratio,error2);
```

```
xlabel('$\frac{Mass}{Theoretical Moment of Inertia}$','interpreter','latex')
ylabel('Error %')
title('Error Proprotional to Mass/Moment of Inertia Ratio')
hold all
plot(645.17,1.25,'.')
plot(645.17,-1.29,'.')
plot(924.69,-.5,'.')
plot(2406,13.71,'.')

legend('Upper Error Bound','Lower Error Bound','1 Wood Block','2 Wood Blocks','ABS Block
```

18 Appendix D: MATLAB Friction Analysis Code

This code simulates the effect of friction. It was used used to determine that friction has a minimal effect on the period for most situations and that it could be safely ignored.

```
%Paul Titchener
%Dynamics Fall 2013 Problem Set 2
%Problem 4: Roller Coaster


function res = roller()
%parameters
g = 9.81

%Initial ConVxditions

theta = .26
thetad = .00001
 %options = odeset('RelTol',1e-1);


[t,Y] = ode45(@rollerstates, [0,10],[theta, thetad]);


figure(2)
hold all
plot(t,Y(:,1))
[pks,locs] = findpeaks(Y(:,1))

time = t(locs)
clear period
for i = 2:4
    period(i-1) = time(i)-time(i-1)
end
avgper = mean(period)

end

function res = rollerstates(t,Z)
r = .05715;
g = 9.8;
l = .88185;
Mp = .0583;
Ip = 1.71655*10^-4; %as found by SOLIDWORKS
```

```
Ip2 =  .00012;%1.0765e-04; %Version 2        6.4088*10^-5 %As derived experimentally
A = .0001;

thetad = Z(2);
%if Z(2) == 0
  % thetadd = (-r^2*Mp*g/l*Z(1))/Ip;  %-A*Mp*Z(1)/abs(Z(1))
%else
thetadd = (-A*Mp*Z(2)/abs(Z(2))-r^2*Mp*g/l*Z(1))/Ip2;  %-A*Mp*Z(1)/abs(Z(1))
%end

res = [thetad;thetadd];
t;

end
```